CHAPTER 3

# POSITIVE SEMIGROUPS FOR TRANSPORT EQUATIONS

The time evolution describing the motion of neutrons in an absorbing and scattering homogeneous medium is given by the following integrodifferential equation

$$\frac{\partial}{\partial t}u(t,x,v) = -v\cdot\nabla_x u(t,x,v) - \sigma(x,v)u(t,x,v) + \int_V \kappa(x,v,v')u(t,x,v')\,dv', \quad (3.1)$$

where $u(t,x,v)$ represents the density distribution of the neutrons in terms of the variables of space $x \in D \subseteq \mathbb{R}^n$ and velocity $v \in V \subseteq \mathbb{R}^n$, at time $t$. Here $D$ denotes the set describing the interior of the vessel in which neutron transport takes place. The medium $D$ is to be thought surrounded by a total absorber (or by a vacuum if $D$ is convex), and neutrons migrate in this volume, are scattered and absorbed by this material. We suppose that neutrons do not interact with each other.
The *free streaming term* $-v\cdot \mathrm{grad}_x u$ in (3.1) is responsible for the motion for the particles between collisions with the background material. The second term of the right-hand side of (3.1) corresponds to collisions including absorption, and the third term to scattering of neutrons: particles at the position $x$ with the incoming speed $v'$ generate particles at $x$ with the outgoing speed $v$ and the transition is governed by a scattering kernel $\kappa(x,v,v')$.
The fact that $u(t,\cdot,\cdot)$ should describe a density suggests to require that $u(t,\cdot,\cdot)$ is an element of $L^1(D\times V)$ for all $t \geq 0$. Following this line and introducing the vector-valued function $u(t) := u(t,\cdot,\cdot)$, (3.1) is equivalent to the following abstract Cauchy problem

$$\begin{cases} u'(t) = (A+K_\kappa)u(t) := (A_0 - M_\sigma)u(t) + K_\kappa u(t), \quad t \geq 0,\\ u(0) \in D(A+K_\kappa). \end{cases}$$

Here $u(t), t \geq 0$, is an element of $L^1(D\times V)$ and $A_0$ denotes the *free streaming opertor* $-v\cdot\mathrm{grad}_x$ on a suitable domain. We refer to [29, Theorem 1.11, p. 36] for a precise description of the domain of $A_0$. Moreover $M_\sigma$ is the multiplication operator by $\sigma$ and is called the *absorption operator*. The *scattering operator* $K_\kappa$ is defined by

$$(K_\kappa f)(x,v) := \int_V \kappa(x,v,v')\, f(x,v')\, dv', \quad (x,v) \in D \times V,\ f \in L^1(D \times V).$$

For more details and information concerning the physical meaning of Equation (3.1) we refer to [34, Chapter 8], (see also [5, Sect. 1.3], [19], [3], [4]).

By using the abstract results given in Section 2.5 we propose to study the asymptotic behaviour of the solution of the transport equation (3.1) (cf. [9, VI.2], [15], [16]). In the first section we present the one-dimensional case and in the second the more general one.

## 3.1 THE ONE-DIMENSIONAL REACTOR PROBLEM

In this section we prove the existence of the semigroup solution of the following transport equation

$$(TE) \qquad \begin{cases} \frac{\partial u}{\partial t} u(t,x,v) = -v\frac{\partial u}{\partial x}(t,x,v) - \sigma(x,v)u(t,x,v) + \\ \qquad\qquad + \int_V \kappa(x,v,v')u(t,x,v')\, dv', \\ \qquad\qquad\qquad t \geq 0,\ (x,v) \in J \times V, \\ u(t,0,v) = 0 \text{ if } v > 0 \text{ and } u(t,1,v) = 0 \text{ if } v < 0, \quad t \geq 0, \\ u(0,x,v) = f(x,v), \quad (x,v) \in J \times V, \end{cases}$$

where $0 \leq \sigma \in L^\infty(J \times V)$, $0 \leq \kappa \in L^\infty(J \times V \times V)$, and $J := [0,1]$, $V := \{v \in \mathbb{R} : v_{min} \leq |v| \leq v_{max}\}$ for given constants $0 < v_{min} < v_{max} < \infty$.
If we suppose that the scattering kernel $\kappa$ satisfies

$$\kappa(x,v,v') > 0 \quad \text{for all } (x,v,v') \in J \times V \times V, \tag{3.2}$$

then one can apply Theorem 2.5.6 and deduce the asymptotic behaviour of the semigroup solution of (TE).
To do so, we recall some results from perturbation theory of $\mathcal{C}_0$–semigroups on Banach spaces.

Let $A$ with domain $D(A)$ be the generator of a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space $E$ and $B \in \mathcal{L}(E)$. Then $A+B$ generates a $\mathcal{C}_0$–semigroup $S(\cdot)$ given by the Dyson-Phillips expansion

$$S(t) = \sum_{n=0}^{\infty} S_n(t), \tag{3.3}$$

where

$$\begin{aligned} S_0(t) &:= T(t) \text{ and} \\ S_{n+1}(t) &:= \int_0^t T(t-s)BS_n(s)\, ds \quad \text{for } x \in E, n \in \mathbb{N} \text{ and } t \geq 0. \end{aligned}$$

The series converges in the operator norm uniformly on bounded intervals of $\mathbb{R}_+$. Some times it is also possible to express the perturbed semigroup $S(\cdot)$ by the Chernoff product formula

$$S(t)x = \lim_{n\to\infty} \Big(T(\frac{t}{n})e^{\frac{t}{n}B}\Big)^n x, \quad t \geq 0, x \in E. \tag{3.4}$$

For these results we refer to [7, III.1], [23, III], [14, I.6] or [9, III].

Recall that an operator $B \in \mathcal{L}(E)$ is called *strictly power compact* if there is $n \in \mathbb{N}$ such that $(BT)^n$ is compact for all $T \in \mathcal{L}(E)$. In particular, if $E$ is an $L^1$–space, then every weakly compact operator is strictly power compact (cf. [8, Corollary VI.8.13]). The following theorem gives the relationship between the essential spectrum of the perturbed and the unperturbed semigroups (see [28] or [9, Theorem IV.4.4]).

**Theorem 3.1.1** *Let $A$ be the generator of a $\mathcal{C}_0$–semigroup $T(\cdot)$ on a Banach space $E$ and $B \in \mathcal{L}(E)$. Let $S(\cdot)$ the $\mathcal{C}_0$–semigroup generated by $A+B$. Assume that there exists $n \in \mathbb{N}$ and a sequence $(t_k) \subset \mathbb{R}_+$, $t_k \to \infty$, such that the remainder $R_n(t_k) := \sum_{p=n} S_p(t_k)$ of the Dyson-Phillips (3.3) at $t_k$ is strictly power compact for all $k \in \mathbb{N}$. Then*

$$r_{ess}(S(t)) \leq r_{ess}(T(t)), \quad t \geq 0.$$

We now give a short description of a special class of regular operators. We denote The *center* of $E$ by

$$\mathcal{Z}(E) := \{M \in \mathcal{L}(E) : MI \subset I \text{ for every closed ideal } I \subset E\},$$

where $E$ is a Banach lattice. It is known that

$$M \in \mathcal{Z}(E) \Longleftrightarrow \pm M \leq \|M\| Id. \tag{3.5}$$

From (3.5) one can see that $(e^{\pm tM})_{t\geq 0}$ is a positive $\mathcal{C}_0$–semigroup whenever $M \in \mathcal{Z}(E)$.

If $(\Omega, \Sigma, \mu)$ is a $\sigma$–finite measure space, then the center $\mathcal{Z}(L^p(\mu))$ is isomorphic to $L^\infty(\mu)$ with the isomorphism

$$L^\infty(\mu) \ni \varphi \mapsto T_\varphi f = \varphi f.$$

To check the irreducibility of the solution semigroup of (TE) we need the following result.

**Proposition 3.1.2** *Let $A_0$ with domain $D(A_0)$ be the generator of a positive $\mathcal{C}_0$–semigroup $T_0(\cdot)$ on a Banach lattice $E$ and $0 \leq K \in \mathcal{L}(E)$. Assume that $0 \leq M \in \mathcal{Z}(E)$. Let $S(\cdot)$ (resp. $T(\cdot)$) be the positive $\mathcal{C}_0$–semigroup generated by $A_0 - M + K$ (resp. $A_0 - M$). If $I \subseteq E$ is a closed ideal, then the following assertion are equivalent.*

(a) *$I$ is $S(\cdot)$–invariant.*

(b) *$I$ is invariant both under $T_0(\cdot)$ and $K$.*

**Proof:** (a) $\Longrightarrow$ (b) Suppose that $I$ is $S(\cdot)$–invariant. Since $0 \leq T(t) \leq S(t)$, $t \geq 0$, it follows that $I$ is $T(\cdot)$–invariant. On the other hand the assumption on $M$, the closedness of $I$ and the formula $e^{tM} = \sum_{n=0}^{\infty} \frac{t^n}{n!} M^n$ imply that $I$ is $e^{tM}$–invariant for all $t \geq 0$. Now, from the product formula (3.4)

$$T_0(t)x = \lim_{n\to\infty} \left( T(\frac{t}{n}) e^{\frac{t}{n}M} \right)^n, \quad t \geq 0, x \in E,$$

we obtain that $I$ is $T_0(\cdot)$–invariant. By (3.3) we have

$$\lim_{t\downarrow 0} \frac{1}{t} \left(S(t)x - T(t)x\right) = \lim_{t\downarrow 0} \frac{1}{t} \int_0^t T(t-s)KS(s)x\, ds = Kx$$

for $x \in E$. Since $I$ is closed and invariant both under $S(\cdot)$ and $T(\cdot)$, we obtain that $I$ is $K$–invariant.

(b) $\Longrightarrow$ (a) It is easy to see that $0 \leq T(t) \leq T_0(t)$, $t \geq 0$. Thus, $I$ is also $T(\cdot)$–invariant. Now, by applying the product formulas (3.4) to $T(t)$ and $e^{tK}$, $t \geq 0$, and using the closedness of $I$, we obtain (a). $\square$

We now return to the transport equation (TE) and define the *free streaming* operator $A_0$ by

$$\begin{aligned}
(A_0 f)(x,v) &:= -v \frac{\partial f}{\partial x}(x,v) \text{ with} \\
D(A_0) &:= \left\{ f \in L^1(J\times V) : v\frac{\partial f}{\partial x} \in L^1(J\times V), \begin{array}{ll} f(0,v) = 0 & \text{if } v > 0 \\ f(1,v) = 0 & \text{if } v < 0 \end{array} \right\},
\end{aligned}$$

the *absorption* operator

$$(M_\sigma f)(x,v) := \sigma(x,v) f(x,v), \quad (x,v) \in J\times V,\ f \in L^1(J\times V),$$

and the *scattering* operator

$$(K_\kappa f)(x,v) := \int_V \kappa(x,v,v') f(x,v')\, dv', \quad (x,v) \in J\times V,\ f \in L^1(J\times V).$$

Let us study first the free streaming operator. By an easy computation one can see that $(0,\infty) \subseteq \rho(A_0)$ and

$$\left(R(\lambda,A_0)f\right)(x,v) = \begin{cases} \frac{1}{v}\int_0^x e^{-\frac{\lambda}{v}(x-x')} f(x',v)\, dx' & \text{if } v > 0, \\ -\frac{1}{v}\int_x^1 e^{-\frac{\lambda}{v}(x-x')} f(x',v)\, dx' & \text{if } v < 0, \end{cases} \tag{3.6}$$

for $(x,v) \in J\times V$ and $f \in L^1(J\times V)$. Hence,

$$(0,\infty) \subseteq \rho(A_0) \text{ and } \|R(\lambda,A_0)\| \leq \frac{1}{\lambda} \quad \text{for all } \lambda > 0.$$

Therefore, by the Hille-Yosida generation theorem (cf. [9, Theorem II.3.5]), $A_0$ with domain $D(A_0)$ generates a $\mathcal{C}_0$–semigroup $T_0(\cdot)$ of contractions on $L^1(J\times V)$. Moreover, $T_0(\cdot)$ is positive since $R(\lambda,A_0)\geq 0$ for all $\lambda>0$. On the other hand, one deduces that

$$(R(\lambda,A_0)f)(x,v)=\int_0^\infty e^{-\lambda t}\chi_J(x-vt)f(x-vt)\,dt$$

for $(x,v)\in J\times V$, $f\in L^1(J\times V)$, where $\chi_J(x)=\begin{cases}1 & \text{if } x\in J,\\ 0 & \text{if } x\notin J.\end{cases}$
So, by the uniqueness of the Laplace transform, we obtain

$$(T_0(t)f)(x,v)=\chi_J(x-tv)f(x-tv,v),\quad (x,v)\in J\times V,\ f\in L^1(J\times V). \tag{3.7}$$

Moreover, since the absorption operator $M_\sigma$ is bounded, it follows that

$$A:=A_0-M_\sigma \text{ with } D(A)=D(A_0)$$

generates the positive $\mathcal{C}_0$–semigroup $T(\cdot)$ given by

$$(T(t)f)(x,v)=e^{-\int_0^t\sigma(x-\tau v,v)\,d\tau}(T_0(t)f)(x,v), \tag{3.8}$$

for $(x,v)\in J\times V$, $f\in L^1(J\times V)$. The boundedness and the positivity of the scattering operator $K_\kappa$ implies that the transport operator $A+K_\kappa$ with domain $D(A_0)$ generates the positive $\mathcal{C}_0$–semigroup $S(\cdot)$ given by the Dyson-Phillips expansion (3.3). This semigroup will be called the *transport semigroup* and satisfies the following properties.

**Proposition 3.1.3** *The streaming semigroup $T(\cdot)$ and the transport semigroup $S(\cdot)$ satisfy*

$$0\leq T(t)\leq S(t)\quad \textit{for all } t\geq 0 \textit{ and} \tag{3.9}$$
$$\omega_0(A+K_\kappa)=s(A+K_\kappa).$$

**Proof:** The first assertion follows from the positivity of $K_\kappa$ and the Dyson-Phillips expansion (3.3). The second is a consequence from Theorem 2.4.1.(ii). □

For the study of the asymptotic behaviour of the transport semigroup we need some properties of weakly compact operators on $L^1$–spaces (see [15, Proposition 2.1] and the references therein).

**Proposition 3.1.4** *Let $(\Omega,\Sigma,\mu)$ be a $\sigma$–finite, positive measure space and $S$, $T$ be two bounded linear operator on $L^1(\Omega,\mu)$. Then the following assertions hold.*

(a) *The set of all weakly compact operators is a norm-closed subset of $\mathcal{L}(L^1(\Omega,\mu))$.*

(b) *If $T$ is weakly compact and $0\leq S\leq T$, then $S$ is also weakly compact.*

(c) *If $S$ and $T$ are weakly compact, then $ST$ is compact.*

We now show the weak compactness of the remainder $R_2(t)$ of the Dyson-Phillips series (3.3) and the irreducibility of the transport semigroup $S(\cdot)$.

**Lemma 3.1.5** *For the transport semigroup $S(\cdot)$ defined above the following properties hold.*

(i) *The remainder $R_2(t) := \sum_{n=2}^{\infty} S_n(t)$, $t \geq 0$, of the Dyson-Phillips expansion (3.3) is a weakly compact operator on $L^1(J \times V)$.*

(ii) *If the scattering kernel satisfies (3.2), then the transport semigroup $S(\cdot)$ is irreducible.*

**Proof:** For $0 \leq f \in L^1(J \times V)$ and $t > 0$ we have

$$\begin{aligned}(K_\kappa T(t) K_\kappa f)(x,v) &\leq (K_\kappa T_0(t) K_\kappa f)(x,v) \\ &\leq \|\kappa\|_\infty^2 \int_V \int_V \chi_J(x - tv'') f(x - tv'', v')\, dv''\, dv' \\ &\leq t^{-1} \|\kappa\|_\infty^2 \int_V \int_J f(x', v')\, dx'\, dv'.\end{aligned}$$

Hence

$$K_\kappa T(t) K_\kappa \leq \frac{\|\kappa\|_\infty^2}{t} (1\!\!1 \otimes 1\!\!1), \tag{3.10}$$

where $1\!\!1 \otimes 1\!\!1$ is the bounded linear operator defined by

$$(1\!\!1 \otimes 1\!\!1) f := \left( \int_J \int_V f(x,v)\, dv\, dx \right) 1\!\!1, \quad f \in L^1(J \times V).$$

By using the definition of the terms $S_n(t)$ in the Dyson-phillips series (3.3) one can see that

$$R_{n+1}(t) := \sum_{k=n+1}^{\infty} S_k(t) = \int_0^t T(t-s) K_\kappa R_n(s)\, ds, \quad t \geq 0, n \in \mathbb{N}.$$

In particular, $R_2(t) = \int_0^t \int_0^{t-s_2} T(s_1) K_\kappa T(s_2) K_\kappa S(t - s_1 - s_2)\, ds_1 ds_2$ for $t \geq 0$. Take $t > \varepsilon > 0$ and consider

$$R_{2,\varepsilon}(t) := \int_\varepsilon^t \int_0^{t-s_2} T(s_1) \left( K_\kappa T(s_2) K_\kappa \right) S(t - s_1 - s_2)\, ds_1 ds_2.$$

Then it is easy to verify that

$$\lim_{\varepsilon \to 0} \|R_{2,\varepsilon}(t) - R_2(t)\| = 0 \quad \text{for all } t > 0.$$

On the other hand, it follows from (3.10) that

$$R_{2,\varepsilon}(t) \leq \|\kappa\|_\infty^2 \int_\varepsilon^t \int_0^{t-s_2} \frac{1}{s_2} T(s_1) \circ (1\!\!1 \otimes 1\!\!1) S(t - s_1 - s_2)\, ds_1 ds_2.$$

From the definition of $T_0(\cdot)$ and since $0 \le T(t) \le T_0(t)$, one can see that $T(t) \circ (1\!\!1 \otimes 1\!\!1) \le (1\!\!1 \otimes 1\!\!1)$ for the order in $\mathcal{L}(L^1(J \times V))$. Now, for $0 \le f \in L^1(J \times V)$, and $s_1 + s_2 \le t$, we obtain

$$\begin{aligned} (1\!\!1 \otimes 1\!\!1)S(t-s_1-s_2)f &= \left(\int_J \int_V (S(t-s_1-s_2)f)(x,v)\,dv\,dx\right) 1\!\!1 \\ &\le Me^{\omega(t-s_1-s_2)} \left(\int_J \int_V f(x,v)\,dv\,dx\right) 1\!\!1 \\ &= Me^{\omega(t-s_1-s_2)}(1\!\!1 \otimes 1\!\!1)f, \end{aligned}$$

where $M \ge 1$ and $\omega \in \mathbb{R}$ are such that $\|S(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Consequently,

$$\begin{aligned} R_{2,\varepsilon}(t) &\le M\|\kappa\|_\infty^2 \left(\int_\varepsilon^t \frac{1}{s_2} \int_0^{t-s_2} e^{\omega(t-s_1-s_2)}\,ds_1 ds_2\right)(1\!\!1 \otimes 1\!\!1) \\ &= \frac{M\|\kappa\|_\infty^2}{\omega} \left(\int_\varepsilon^t \frac{e^{\omega(t-s_2)}-1}{s_2}\,ds_2\right)(1\!\!1 \otimes 1\!\!1). \end{aligned}$$

This implies that $R_{2,\varepsilon}(t)$ is dominated by a one-dimensional operator. So, by Proposition 3.1.4, we obtain that $R_{2,\varepsilon}(t)$ is weakly compact and therefore $R_2(t)$ is weakly compact for all $t \ge 0$. This proves (i).
We recall that every closed ideal in $L^1(J \times V)$ has the form

$$I = \{f \in L^1(J \times V) : f \text{ vanish a.e. on } \Omega\}$$

for some measurable subset $\Omega \subseteq J \times V$. We suppose that $I$ is $S(\cdot)$–invariant. Then, by Proposition 3.1.2, $I$ is $K_\kappa$–invariant. Assume that $\Omega \ne \emptyset$. Since $\chi_{J\times V\setminus\Omega} \in I$, we obtain

$$\begin{aligned} \left(K_\kappa \chi_{J\times V\setminus\Omega}\right)(x,v) &= \int_V \kappa(x,v,v')\chi_{J\times V\setminus\Omega}(x,v')\,dv' \\ &= \int_{V\setminus\Omega_x} \kappa(x,v,v')\,dv' = 0 \end{aligned}$$

for $(x,v) \in \Omega$ and $\Omega_x := \{v \in V : (x,v) \in \Omega\}$. Since $\kappa$ is strictly positive, it follows that $\Omega_x = V$. Hence, $\Omega = Y \times V$ for some measurable subset $Y$ of $J$.
On the other hand, again by Proposition 3.1.2, $I$ is $T_0(\cdot)$–invariant. Thus, $I$ is $R(\lambda, A_0)$–invariant for all $\lambda > 0$. Hence, $\left(R(\lambda, A_0)\chi_{J\times V\setminus\Omega}\right)(x,v) = 0$ for a.e. $(x,v) \in \Omega$. So, by using (3.6), one can see that

$$\int_0^x \chi_{J\setminus Y}(s)\,ds = 0 \text{ and } \int_x^1 \chi_{J\setminus Y}(s)\,ds = 0.$$

Therefore, $\int_0^1 \chi_{J\setminus Y}(s)\,ds = 0$ and this implies that $Y = J$. Consequently, $I = \{0\}$ or $I = L^1(J \times V)$ and (ii) is proved. ◻

We can now describe the asymptotic behaviour of the transport semigroup.

**Theorem 3.1.6** *Assume that* $\kappa$ *satisfies (3.2). Then the transport semigroup* $S(\cdot)$ *has balanced exponential growth. More precisely, there are two strictly positive functions* $\varphi \in L^1(J\times V)$ *and* $\psi \in L^\infty(J\times V)$ *satisfying* $\int_{J\times V}\varphi(x,v)\psi(x,v)\,dv\,dx = 1$ *such that*

$$\|e^{-s(A+K_\kappa)t}S(t) - \psi\otimes\varphi\| \leq Me^{-\varepsilon t}$$

*for all* $t \geq 0$ *and some constants* $M \geq 0$ *and* $\varepsilon > 0$.

**Proof:** Since $v_{min} > 0$, it follows that $T(\cdot)$ is a nilpotent semigroup, i.e., there is $t_0 > 0$ such that

$$T(t) = 0 \quad \text{for all } t \geq t_0. \tag{3.11}$$

Hence, $r(T(t)) = r_{ess}(T(t)) = 0$ for all $t > 0$. So, by Lemma 3.1.5.(i) and Theorem 3.1.1, we have

$$\omega_{ess}(A+K_\kappa) = -\infty.$$

On the other hand, it follows from (3.11) that

$$S_1(t) = \int_0^t T(s)K_\kappa T(t-s)\,ds = 0 \quad \text{for all } t \geq 2t_0$$

and therefore

$$R_2(t) = S(t) \quad \text{for all } t \geq 2t_0.$$

So, by Lemma 3.1.5.(ii), we obtain that $R_2(t)$ is irreducible for all $t \geq 2t_0$. Now, one can apply [27, Theorem A.(iii)] to obtain that $r(S(t)) = r(R_2(t)) > 0$ for all $t \geq 2t_0$. Therefore,

$$-\infty = \omega_{ess}(A+K_\kappa) < \omega_0(A+K_\kappa).$$

Then one can apply Theorem 2.5.6 to the transport semigroup $S(\cdot)$ and obtains the assertions. □

## 3.2 THE N-DIMENSIONAL REACTOR PROBLEM

The second example is concerned with the n-dimensional transport equation (see [30] and [31])

$$(nTE) \qquad \begin{cases} \frac{\partial u}{\partial t}u(t,x,v) = -v\cdot\nabla_x u(t,x,v) - \sigma(x,v)u(t,x,v) + \\ + \qquad\qquad\qquad + \int_V \kappa(x,v,v')u(t,x,v')\,dv', \\ \qquad\qquad\qquad\qquad t\geq 0,\ (x,v)\in D\times V, \\ u(t,\cdot,\cdot)|_{\Gamma_-} = 0, \quad t \geq 0, \\ u(0,x,v) = f(x,v), \quad (x,v)\in D\times V \end{cases}$$

on $L^1(D\times V)$, where $\Gamma_- := \{(x,v)\in\partial D\times V : v\cdot n(x) < 0\}$ and $n(x)$ is the outward normal at $x\in\partial D$.

We propose again to apply the theory developed in Section 2.5 to study the asymptotic behaviour of the solution of the transport equation (nTE). Proving the irreducibility of the transport semigroup in this case is not so easy.

We suppose that $D$ is a smooth open subset of $\mathbb{R}^n$ and $V$ is an open subset of $\mathbb{R}^n$. The collision $\sigma$ and the scattering kernel $\kappa$ are nonnegative and measurable functions satisfying

$$\sigma \in L^\infty(D\times V) \text{ and } \sup_{(x,v)\in D\times V}\left(\int_V \kappa(x,v',v)\,dv'\right) < \infty. \tag{3.12}$$

Condition (3.12) implies that the absorption operator $M_\sigma$ and the scattering operator $K_\kappa$ are both bounded on $L^1(D\times V,\mu)$, where $\mu$ is the $2n$-dimensional Lebesgue measure. As in the previous section, we define the free streaming semigroup, the absorption semigroup and the transport semigroup respectively by

$$\begin{aligned}
(T_0(t)f)(x,v) &:= f(x-tv,v)\chi_t(x,v)\\
(T(t)f)(x,v) &:= \exp\left(-\int_{-t}^{0}\sigma(x+sv,v)\,ds\right)(T_0(t)f)(x,v)\\
S(t) &:= \sum_{n=0}^{\infty} S_n(t),
\end{aligned}$$

where $\chi_t(x,v) := \begin{cases} 1 & \text{if } t_-(x,v) > t \\ 0 & \text{if } t_-(x,v) \le t \end{cases}$ and $t_-(x,v) := \inf\{s>0 : x-sv \notin D\}$, $(x,v)\in D\times V$, $S_0(t) = T(t)$ and

$$S_{n+1}(t) = \int_0^t T(t-s)K_\kappa S_n(s)\,ds \text{ for } t\ge 0 \text{ and } (x,v)\in D\times V.$$

If we denote by $A_0$ the generator of $(T_0(t))_{t\ge 0}$, then $A = A_0 - M_\sigma$ and $A+K_\kappa$ are the generator of $T(\cdot)$ and $S(\cdot)$ respectively. We note that those semigroups are positive and strongly continuous on $L^1(D\times V,\mu)$.
In order to illustrate the theory given in Section 2.5, let us consider the special case where

$$\begin{cases} D \text{ is bounded and connected and } \{v\in\mathbb{R}^n : \xi_1 < |v| < \xi_2\} =: V_0 \subset \\ \qquad V \subset V_1 := \{v\in\mathbb{R}^n : |v| > v_{\min}\} \end{cases} \tag{3.13}$$

for some constants $v_{\min} > 0$ and $0 \le \xi_1 < \xi_2 \le \infty$.
Without loss of generality one can suppose that $\xi_2 < \infty$.
As in the previous section, the second order remainder

$$R_2(t) := \sum_{n=2}^{\infty} S_n(t), \quad t\ge 0,$$

of the Dyson-Phillips expansion (3.3) will be of particular importance. If we denote $S_t := \{(s_1,s_2) : s_1,s_2 \ge 0 \text{ and } s_1+s_2 \le t\}$, one can see that

$$R_2(t) = \int_{S_t} T(s_1)K_\kappa T(s_2)K_\kappa S(t-s_1-s_2)\,ds_1\,ds_2$$

holds for $t \geq 0$. In particular, we have

$$\begin{aligned} & (T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) \\ = \; & \sigma_{s_1}(x,v)\int_V \kappa(x-s_1v,v,v'')\sigma_{s_2}(x-s_1v,v'') \\ & \cdot \int_V \kappa(x-s_1v-s_2v'',v'',v')\, f(x-s_1v-s_2v'',v')\,dv'\,dv'' \end{aligned}$$

for $f \in L^1(D\times V)$, where

$$\sigma_s(x,v) := \chi_s(x,v)\exp\left(-\int_{-s}^0 \sigma(x+\tau v,v)\,d\tau\right)$$

for $(x,v) \in D\times V$. By taking the new variable $x' := x-s_1v-s_2v''$ we obtain

$$(T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) = \int_{D\times V} \tilde{\kappa}_{s_1,s_2}(x,v,x',v')\, f(x',v')\,dx'dv',$$

where

$$\begin{aligned} & \tilde{\kappa}_{s_1,s_2}(x,v,x',v') && (3.14) \\ := \; & \sigma_{s_1}(x,v)s_2^{-n}\kappa\left(x-s_1v,v,\frac{x-x'-s_1v}{s_2}\right) \\ & \cdot\sigma_{s_2}\left(x-s_1v,\frac{x-x'-s_1v}{s_2}\right)\kappa\left(x',\frac{x-x'-s_1v}{s_2},v'\right). && (3.15) \end{aligned}$$

Here and in the sequel we use the convention that all functions defined on $D\times V$ ($D\times V\times V$ resp.) are extended by zero to $\mathbb{R}^n\times\mathbb{R}^n$ (resp. $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n$). If we suppose that $\kappa$ satisfies the conditions

$$\exists\gamma\in L^1(V)\,/\,\kappa(x,v,v') \leq \gamma(v) \text{ for all } (x,v,v')\in D\times V\times V \qquad (3.16)$$

and

$$V_0 \subset V \text{ and } \kappa(\cdot,\cdot,\cdot) > 0 \text{ on } (D\times V_0\times V)\cup(D\times V\times V_0), \qquad (3.17)$$

then we have the main result of this section.

**Theorem 3.2.1** *Suppose that* (3.12) *and* (3.13) *hold. If* $\kappa$ *satisfies the conditions* (3.16) *and* (3.17)*, then there exist* $0 < \varphi \in L^1(D\times V)$, $0 < \psi \in L^\infty(D\times V)$ *with* $\int_D\int_V \varphi(x,v)\psi(x,v)\,dv\,dx = 1$ *such that*

$$\|e^{-s(A+B)t}S(t) - \psi\otimes\varphi\| \leq Me^{-\varepsilon t}$$

*for all* $t \geq 0$ *and some constants* $M \geq 1$ *and* $\varepsilon > 0$.

The proof is split into two lemmas.

**Lemma 3.2.2** *Suppose that* $D$ *is bounded and* (3.12), (3.16) *are satisfied. Then the second order remainder* $R_2(t)$ *is weakly compact for all* $t > 0$*. Therefore,*

$$r_{ess}(S(t)) \leq r_{ess}(T(t) \quad \textit{for } t \geq 0.$$

**Proof:** Since

$$R_2(t) = \int_{S_t} T(s_1)K_\kappa T(s_2)K_\kappa S(t-s_1-s_2)\,ds_1\,ds_2$$

and by [32, Theorem 1.3], it suffices to show that the operators $T(s_1)K_\kappa T(s_2)K_\kappa$ are weakly compact for all $(s_1,s_2) \in S_t$ with $s_2 > 0$. Let us note that we have

$$(T(s_1)K_\kappa T(s_2)K_\kappa f)(x,v) = \int_{D\times V} \tilde{\kappa}_{s_1,s_2}(x,v,x',v')\,f(x',v')\,dx'\,dv'$$

with $\tilde{\kappa}_{s_1,s_2}$ from (3.14). It follows from the Dunford-Pettis theorem (cf. [8, Theorem 10, p. 507]) that it suffices to prove that the set

$$M := \{\tilde{\kappa}_{s_1,s_2}(\cdot,\cdot,x',v');\ (x',v') \in D\times V\}$$

is contained in a weakly compact subset of $L^1(D\times V,\mu)$. We note that the function

$$D \ni x' \mapsto g(x') \in L^1(D\times V,\mu)$$

defined by

$$g(x')(x,v) := s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1 v}{s_2}\right), \quad (x,v)\in D\times V,$$

is continuous. This statement follows from a simple estimate by approximating $\gamma \in L^1(V)$ by continuous functions with compact support. So, since $D$ is bounded, it follows that the set

$$\tilde{M} := \{s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1 v}{s_2}\right) : x' \in \overline{D}\}$$

is relatively compact in $L^1(D\times V,\mu)$. By (3.16) we now have

$$0 \le \tilde{\kappa}_{s_1,s_2}(x,v,x',v') \le s_2^{-n}\gamma(v)\gamma\left(\frac{x-x'-s_1 v}{s_2}\right)$$

for $(x,v,x',v') \in (D\times V)\times(D\times V)$. Therefore, the Dunford-Pettis theorem (cf. [21, Theorem 2.5.4.(iv)]) implies that $M$ is relatively weakly compact in $L^1(D\times V,\mu)$. The last assertion follows from Theorem 3.1.1. $\square$

**Lemma 3.2.3** *Assume that $D$ is connected and* (3.12) *is satisfied. Let $V_0$ be the set given in* (3.13)*. If* (3.17) *holds, then $(S(t))_{t\ge 0}$ is irreducible.*

**Proof:** **1.** Let us prove first that, for $x_0 \in D$ and $r > 0$ such that

$$B(x_0,3r) := \{x \in \mathbb{R}^n : |x_0 - x| < 3r\} \subset D,$$

we have for each $0 \leq f \in L^1(D \times V, \mu)$ with $f|_{B(x_0,r)\times V} \neq 0$

$$\left(S(\frac{2r}{\xi_2}) f\right)(x,v) > 0 \text{ for a.e. } (x,v) \in B(x_0,r) \times V. \tag{3.18}$$

To this purpose let us consider the second order term $S_2(\cdot)$ of the Dyson-Phillips series (3.3) and put $t_0 := \frac{2r}{\xi_2}$. Then by a simple calculation one can see that

$$\begin{aligned}(S_2(t_0) f)(x,v) &= \left(\int_{S_{t_0}} T(s_1)K_\kappa T(t_0-s_1-s_2)K_\kappa T(s_2) f\, ds_1 ds_2\right)(x,v)\\ &= \int_{D\times V} \beta(x,v,x',v') f(x',v')\, dx' dv',\end{aligned}$$

where

$$\begin{aligned}&\beta(x,v,x',v')\\ := &\int_{S_{t_0}} \sigma_{s_1}(x,v)\sigma_{t_0-s_1-s_2}\left(x-s_1 v, \frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\cdot\sigma_{s_2}(x'+s_2v',v')(t_0-s_1-s_2)^{-n}\kappa\left(x-s_1v, v, \frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\quad\cdot\kappa\left(x'+s_2v', \frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}, v'\right) ds_2\, ds_1,\\ = &\int_{S_{t_0}} \sigma_{s_1}(x,v)\sigma_{t_0-s_1-s_2}\left(x-s_1v, \frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}\right)\\ &\cdot\sigma_{s_2}(x'+s_2v',v')(t_0-s_1-s_2)^{-n}\\ &\cdot\tilde\beta\left(x-s_1v, x'+s_2v', v, \frac{x-s_1v-x'-s_2v'}{t_0-s_1-s_2}, v'\right) ds_2\, ds_1,\end{aligned}$$

with $\tilde\beta : D\times D\times V\times V\times V \to [0,\infty)$ given by

$$\tilde\beta(x,x',v,v',v'') := \kappa(x,v,v')\kappa(x',v',v'').$$

From (3.17) we know that $\tilde\beta(\cdot,\cdot,\cdot,\cdot,\cdot) > 0$ on $B(x_0,3r)\times B(x_0,3r)\times V\times V_0\times V$. Now, for a.e. $(x,v,x',v') \in B(x_0,r)\times V\times B(x_0,r)\times V$, it follows from Exercise 3.2.4 below that $\beta(x,v,x',v') > 0$. Therefore, since $0 \leq S_2(t_0) \leq S(t_0)$, we obtain the first assertion.

**2.** The claim given in (3.18) holds for all $t \geq t_0$. In fact, choose $m \in \mathbb{N}$ such that $\frac{t-t_0}{m} \leq t_0$ and instead of $r$ we take $r' := \frac{\xi_2(t-t_0)}{2m}$ $(\leq r)$. Then, (3.18) can be applied $m$ times to each ball $B(x_0,r')$ contained in $B(x_0,r)$ and we obtain

$$(S(t)f)(x,v) = S\left(\frac{t-t_0}{m}\right)^m (S(t_0)f)(x,v) > 0$$

for a.e. $(x,v) \in B(x_0,r')\times V$ and for all $0 \leq f \in L^1(D\times V,\mu)$ such that $f|_{B(x_0,r)\times V} \neq 0$. Consequently, $(S(t)f)(x,v) > 0$ for a.e. $(x,v) \in B(x_0,r)\times V$.

**3.** Finally we show that $(S(t))_{t\geq 0}$ is irreducible. Let $0 \lneqq f \in L^1(D\times V,\mu)$. Then there is $x_0 \in D$ such that for all $\varepsilon > 0$ with $B(x_0,\varepsilon) \subset D$ and $f|_{B(x_0,\varepsilon)\times V} \neq 0$. Let $t > 0$ and consider $x' \in D$ such that there exists a polygonal path $C$ of length $< \xi_2 t$ connecting $x_0$ with $x'$. There exists a covering of $C$ by balls $(B(x_i,r_i))_{i=0,\ldots,m}$ such that $x_m = x'$, $B(x_i,r_i)\cap B(x_{i-1},r_{i-1}) \neq \emptyset$ for $i = 1,\ldots,m$, $B(x_i,3r_i) \subset D$ for $i = 0,\ldots,m$ and $2\sum_{i=0}^m r_i \leq \xi_2 t$. If we repeat this procedure we have

$$(S(t)f)(x,v) > 0 \qquad \text{for a.e. } (x,v) \in B(x',r_m)\times V,$$

and the lemma is proved. $\square$

**Proof of Theorem 3.2.1** From (3.13) we have in particular

$$V \subset V_1 := \{v \in \mathbb{R}^n; |v| > v_{min}\}.$$

This and the boundedness of $D$ imply that there is $t_0 > 0$ such that $T_0(t) = T(t) = 0$ for all $t \geq t_0$ and therefore, $\omega_0(A_0) = \omega_0(A) = -\infty$. Thus

$$\int_0^t T(t-s)K_\kappa T(s)\,ds = 0 \qquad \text{for all } t \geq 2t_0.$$

This implies,

$$S(t) = R_2(t) \qquad \text{for all } t \geq 2t_0.$$

So, by Lemma 3.2.2 and Lemma 3.2.3, $(S(t))_{t\geq 0}$ is irreducible and consists of weakly compact operators for all $t \geq 2t_0$. Hence, it follows from [27, Theorem A] that $\omega_0(A+K_\kappa) > \omega_{ess}(A+K_\kappa)\,(= -\infty)$. Now, the result follows from Theorem 2.5.6. $\square$

**Exercise 3.2.4** *Use the notation from the proof of Lemma 3.2.3 and define the function* $\alpha : D\times V\times D\times V\times S_{t_0} \to \mathbb{R}^{5n}$,

$$\alpha(x,v,x',v',s_1,s_2) := \left(x - s_1 v, x' + s_2 v', v, \frac{x - s_1 v - x' - s_2 v'}{t_0 - s_1 - s_2}, v'\right).$$

*Show that, for a.e.* $(x,v,x',v') \in B(x_0,r)\times V\times B(x_0,r)\times V$, *the set*

$$\{(s_1,s_2) \in S_{t_0}; \alpha(x,v,x',v',s_1,s_2) \in B(x_0,3r)\times B(x_0,3r)\times V\times V_0\times V\}$$

*is open and nonempty.*